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MODIFIED FLEXIBLE WEIBULL DISTRIBUTION

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ABSTRACT

This article is devoted to propose a new five-parameter novel probability model, named as Modified Flexible Weibull distribution. The proposed model is a mixture of two components: first component is a flexible Weibull model and another component is exponential distribution, and exhibits bathtub-shaped failure rate. Some of the mathematical properties of the proposed model, including moments, generating functions, quantile and order statistics are derived. The model parameters are estimated by the maximum likelihood method. To show the workability of the proposed model, a real life example is analyzed and it is observed that, the proposed model performs better than the prominent lifetime distributions, including: Weibull, flexible Weibull (FW), flexible Weibull extension (FWEx), exponentiated Weibull (EW), exponentiated flexible Weibull extension (EFWEx), exponential flexible Weibull extension (EFWEx), transmuted Weibull (TW), and Kumaraswamy Weibull (Ku-w).

KEYWORDS: Flexible Weibull Distribution, Bathtub Shaped Failure Rates, Moment Generating Function, Order Statistics, Maximum Likelihood Estimates

1. INTRODUCTION

The Weibull model, proposed by Weibull [14] is a familiar and most frequently used distribution, for modeling real phenomena, where the failure rate (hazard rate) is monotonic. But, the Weibull model is inappropriate to use for modeling lifetime data, with non-monotonic failure rate. In the later literature, numerous researchers have shown a deep interest to extend the two-parameter Weibull distribution, to obtain non-monotonic failure rate for modeling lifetime data. In the recent years, new extensions of Weibull model have been proposed, to provide a better fit to data having bathtub-shaped failure rate function. Among these modifications: Beta-Weibull (BW) distribution of Famoye et al. [9], new flexible Weibull (NFW) and new extended Weibull (NEx-W) proposed by Ahmad and Hussain [3] and [7], respectively. Modified Weibull (MW) distribution by Sarhan and Zaindin [12], Beta modified Weibull (BMW) distribution proposed by Silva et al. [13], and Flexible Weibull extension (FWEx) distribution by Bebbington et al. [8], etc. A detailed review of these modifications is summarized in Pham and Lai [11] and Murthy et al. [10]. Recently, Ahmad and Hussain [2], proposed a new extension of the Weibull distribution, called flexible Weibull (FW) distribution. The cumulative distribution function (CDF) of the FW distribution is given by

$$G(z;\alpha,\gamma,\beta,\theta)=1-e^{-e^{\left(\beta z^{\gamma}+\theta z^{\alpha}\right)}}, \qquad z,\alpha,\gamma,\beta,\theta>0. \tag{1}$$

The probability density function (PDF) of FW model is given by

$$g(z;\alpha,\gamma,\beta,\theta)=\left(\gamma\beta z^{\gamma-1}+\alpha\theta z^{\alpha-1}\right)e^{\left(\beta z^{\gamma}+\theta z^{\alpha}\right)}\,e^{-e^{\left(\beta z^{\gamma}+\theta z^{\alpha}\right)}}. \tag{2}$$

In the present article, a new extension of the FW distribution called modified flexible Weibull (MFW) distribution is derived. Using the generator $-\log\left(1-G\left(z\right)\right)$, Ahmad and Iqbal [4] and Ahmad and Hussain [6], proposed modified new flexible Weibull (MNFW) and generalized flexible Weibull extension (GFWEx) distributions, respectively. For a brief review about such type of generalizations, see Ahmad and Hussain [5]. If *G* (z) is the CDF of a baseline random variable, with PDF *g* (*z*) and the exponential CDF is

$$F\left(z\right)=1-e^{-\lambda z}\ , \qquad z,\lambda>0.$$

Based on this density, by replacing *z* with $-\log\left(1-G\left(z\right)\right)$, extensions of the Weibull model are defined by (see Ahmad and Iqbal [4] and Ahmad and Hussain [6]).

$$F\left(z\right)=\int_{0}^{-\log\left(1-G\left(z\right)\right)}\lambda e^{-\lambda z}\ dz,$$

$$F\left(z\right)=1-\left(1-G(z)\right)^{\lambda}. \tag{3}$$

The density corresponding to (3), given by

$$F\left(z\right)=\lambda g\left(z\right)\left(1-G(z)\right)^{\lambda-1}. \tag{4}$$

By using the CDF and PDF of FW distribution, in (3), and in (4), one may have the CDF and PDF of the proposed model. The proposed model is named as modified flexible Weibull (MFW) distribution, and is able to model life time data with bathtub-shaped failure rates. The present paper is structured as: Section 2, offers the definition and graphical display of the proposed model. Section 3, provides the basic statistical properties. Section 4, 5, and 6, derives the moment generating, probability moment generating, and factorial moment generating functions of the proposed distribution, resectively. Section 7 and 8, contains the estimation of the model parameters and densities of the order statistics, respectively. Section 9, provides the analysis of a real data set. Finally, section 10; draw the conclusion of the article.

2. MODIFIED FLEXIBLE WEIBULL DISTRIBUTION

The distribution function of the MFW distribution is given by

$$F\left(z;\alpha,\gamma,\beta,\theta,\lambda\right)=1-\ e^{-\lambda e^{\left(\beta z^{\gamma}\ +\ \theta z^{\alpha}\right)}}, \qquad z,\alpha,\gamma,\beta,\theta,\lambda>0. \tag{5}$$

The density function corresponding to (5) is given by

$$f\left(z;\alpha,\gamma,\beta,\theta,\lambda\right)=\lambda\left(\gamma\beta z^{\gamma-1}+\alpha\theta z^{\alpha-1}\right)e^{\left(\beta z^{\gamma}\ +\ \theta z^{\alpha}\right)}\ e^{-\lambda e^{\left(\beta z^{\gamma}\ +\ \theta z^{\alpha}\right)}}.$$

The survival function (SF) of the MFW distribution is

$$S\left(z;\alpha,\gamma,\beta,\theta,\lambda\right)=\ e^{-\lambda e^{\left(\beta z^{\gamma}\ +\ \theta z^{\alpha}\right)}},$$

With hazard function (HF) given by

$$h\left(z;\alpha,\gamma,\beta,\theta,\lambda\right)=\lambda\left(\gamma\beta z^{\gamma-1}+\alpha\theta z^{\alpha-1}\right)e^{\left(\beta z^{\gamma}\ +\ \theta z^{\alpha}\right)}. \tag{6}$$

The figure 1 displays the HF's of the MFW distribution for different values of parameters.

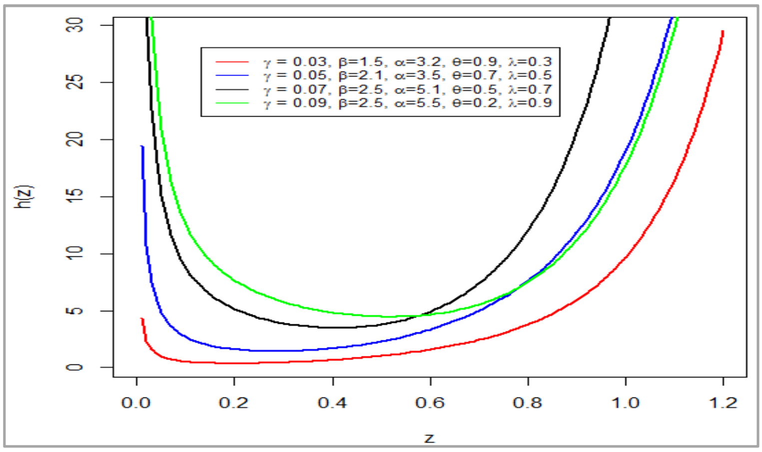

Figure 1: HF of the Modified Flexible Weibull Distribution, for Different Values of Parameters

3. BASIC PROPERTIES

This section of the article provides the basic mathematical properties of the MFW distribution.

3.1 Quantile and Median

The formula for the q^{th} quantile z_q of the MFW distribution is given by

$$\beta z_q^{\gamma} + \theta z_q^{\alpha} - \log\left\{-\frac{\log(1-q)}{\lambda}\right\} = 0. \tag{7}$$

Using $q = 0.50$, in (7), one may easily derive the median of the MFW distribution. Also, using $q = 0.25$, and $q = 0.75$, in (7), one may have the *1st* and *3rd* quartiles of the proposed model, respectively.

3.3 Generation of Random Numbers

The formula for generating random numbers from MFW model is provided below

$$\beta z^{\gamma} + \theta z^{\alpha} - \log\left\{-\frac{\log(1-R)}{\lambda}\right\} = 0, \qquad R \sim U(0,1).$$

3.3 Moments

If Z~MFWD$(z;\alpha,\gamma,\beta,\theta,\lambda)$, then the r^{th} moments of Z is derived as

$$\mu_r^{/} = \int_0^{\infty} z^r \left(z;\alpha,\gamma,\beta,\theta,\lambda\right) dz,$$

$$\mu_r^{/} = \int_0^{\infty} z^r \, \lambda\left(\gamma\beta z^{\gamma-1} + \alpha\theta z^{\alpha-1}\right) e^{\left(\beta z^{\gamma} + \theta z^{\alpha}\right)} \, e^{-\lambda e^{\left(\beta z^{\gamma} + \theta z^{\alpha}\right)}} dz,$$

$$\mu_r^{/}=\sum_{i=0}^{\infty}\frac{(-1)^i\lambda^{i+1}}{i!}\left\{\int_0^{\infty}z^r\left(\gamma\beta z^{\gamma-1}+\alpha\theta z^{\alpha-1}\right)\left(e^{\left(\beta z^{\gamma}+\theta z^{\alpha}\right)}\right)^{i+1}\right\}dz,$$

$$\mu_r^{/}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i(i+1)^j\lambda^{i+1}\beta^j}{i!\,j!}\left\{\int_0^{\infty}z^{j\gamma+r}\left(\gamma\beta z^{\gamma-1}+\alpha\theta z^{\alpha-1}\right)e^{\theta(i+1)z^{\alpha}}\right\}dz,\text{s}$$

$$\mu_r^{/}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i(i+1)^j\beta^j\lambda^{i+1}}{i!\,j!}\left\{\gamma\beta\int_0^{\infty}z^{r+\gamma(j+1)-1}\,e^{\theta(i+1)z^{\alpha}}\,dz+\alpha\theta\int_0^{\infty}z^{r+j\gamma+\alpha-1}\,e^{\theta(i+1)z^{\alpha}}\,dz\right\}, \quad \textbf{(8)}$$

Using the definition of gamma function, see Zwillinger [15] in the following form,

$$\Gamma z=x^z\int_0^{\infty}t^{z-1}e^{tx}dt,\qquad z,x>0.$$

Using the above definition of gamma function in (8), finally, we get

$$\mu_r^{/}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i(i+1)^j\lambda^{i+1}\beta^j}{i!\,j!}\left\{\gamma\beta\frac{\Gamma\left(\frac{\gamma(j+1)+r}{\alpha}\right)}{\alpha\left(\theta(i+1)\right)^{\frac{\gamma(j+1)+r}{\alpha}}}+\theta\frac{\Gamma\left(\frac{j\gamma+r+1}{\alpha}\right)}{\left(\theta(i+1)\right)^{\frac{j\gamma+r+1}{\alpha}}}\right\}. \quad \textbf{(9)}$$

4. MOMENT GENERATING FUNCTION

By definition the moment generating function (MGF) can be derived as

$$M_z(t)=\int_0^{\infty}e^{tz}g(z;\alpha,\gamma,\beta,\theta,\lambda)\,dz$$

$$M_z(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\int_0^{\infty}z^r g(z;\alpha,\gamma,\beta,\theta,\lambda)\,dz$$

$$M_z(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r^{/}. \quad \textbf{(10)}$$

By using (9), in (10), one might get the proof of the MGF of MFW distribution.

5. PROBABILITY GENERATING FUNCTION

The probability generating function (PGF) of MFW model is derived as

$$G(\gamma)=\int_0^{\infty}\gamma^z g(z;\alpha,\gamma,\beta,\theta,\lambda)\,dz$$

$$G(\gamma)=\sum_{r=0}^{\infty}\frac{\log^r(\gamma)}{r!}\int_0^{\infty}z^r g(z;\alpha,\gamma,\beta,\theta,\lambda)\,dz,$$

$$G(\gamma)=\sum_{r=0}^{\infty}\frac{\log^{r}(\gamma)}{r!}\mu_r^{/}. \tag{11}$$

On substituting (9), in (11), one may obtain the expression for the PGF of MFW model.

6. FACTORIAL MOMENT GENERATING FUNCTION

The factorial moment generating function (FMGF) of MFW distribution can be obtained as

$$H_0(1+\delta)=\int_0^{\infty}(1+\delta)^{z}\,g(z;\alpha,\gamma,\beta,\theta,\lambda)\,dz$$

$$H_0(1+\delta)=\sum_{r=0}^{\infty}\frac{\log^{r}(1+\delta)}{r!}\int_0^{\infty}z^{r}g(z;\alpha,\gamma,\beta,\theta,\lambda)\,dz,$$

$$H_0(1+\delta)=\sum_{r=0}^{\infty}\frac{\log^{r}(1+\delta)}{r!}\mu_r^{/}. \tag{12}$$

By substituting (9), in (12), one may get the proof of the FMGF of MFW distribution.

7. ESTIMATION

This section of the article, which concern with estimation of the model parameters, through maximum likelihood (ML) procedure. Let $Z_1, Z_2, \ldots, Z_k$, be a MFW distribution led from MFW distribution, with parameters $(\alpha,\gamma,\beta,\theta,\lambda)$, the corresponding log-likelihood function of this sample is given by

$$\ln L = k\log\lambda+\sum_{i=1}^{k}\log\left(\gamma\beta z_i^{\gamma-1}+\alpha\theta z_i^{\alpha-1}\right)+\sum_{i=1}^{k}\left(\beta z_i^{\gamma}+\theta z_i^{\alpha}\right)-\lambda\sum_{i=1}^{k}e^{\left(\beta z_i^{\gamma}+\theta z_i^{\alpha}\right)}. \tag{13}$$

By obtaining the partial derivatives of the expression in (13) on a parameter, and then equating the derived result to zero, one may get

$$\frac{d\ln L}{d\beta}=\sum_{i=1}^{k}\frac{\gamma z_i^{\gamma-1}}{\left(\gamma\beta z_i^{\gamma-1}+\alpha\theta z_i^{\alpha-1}\right)}+\sum_{i=1}^{k}z_i^{\gamma}-\lambda\sum_{i=1}^{k}z_i^{\gamma}e^{\left(\beta z_i^{\gamma}+\theta z_i^{\alpha}\right)}. \tag{14}$$

$$\frac{d\ln L}{d\theta}=\sum_{i=1}^{k}\frac{\alpha z_i^{\alpha-1}}{\left(\gamma\beta z_i^{\gamma-1}+\alpha\theta z_i^{\alpha-1}\right)}+\sum_{i=1}^{k}z_i^{\alpha}-\lambda\sum_{i=1}^{k}z_i^{\alpha}e^{\left(\beta z_i^{\gamma}+\theta z_i^{\alpha}\right)}. \tag{15}$$

$$\frac{d\ln L}{d\alpha}=\theta\sum_{i=1}^{k}\frac{\left(\alpha z_i^{\alpha-1}\log(z_i)+z_i^{\alpha-1}\right)}{\left(\gamma\beta z_i^{\gamma-1}+\alpha\theta z_i^{\alpha-1}\right)}+\theta\sum_{i=1}^{k}z_i^{\alpha}\log(z_i)-\theta\lambda\sum_{i=1}^{k}z_i^{\alpha}\log(z_i)e^{\left(\beta z_i^{\gamma}+\theta z_i^{\alpha}\right)}. \tag{16}$$

$$\frac{d\ln L}{d\gamma}=\beta\sum_{i=1}^{k}\frac{\left(\gamma z_i^{\gamma-1}\log(z_i)+z_i^{\gamma-1}\right)}{\left(\gamma\beta z_i^{\gamma-1}+\alpha\theta z_i^{\alpha-1}\right)}+\beta\sum_{i=1}^{k}z_i^{\gamma}\log(z_i)-\beta\lambda\sum_{i=1}^{k}z_i^{\gamma}\log(z_i)e^{\left(\beta z_i^{\gamma}+\theta z_i^{\alpha}\right)}. \tag{17}$$

$$\frac{d\ln L}{d\lambda}=\frac{k}{\lambda}-\sum_{i=1}^{k}e^{\left(\beta z_i^{\gamma}+\theta z_i^{\alpha}\right)}\ . \quad \textbf{(18)}$$

It is observed that, the expressions provides in (14)-(18) do not possess a closed form solution; so, the estimates of the unknown parameters can be obtained numerically by using the iterating procedure. The “SANN” algorithm in R language is deployed to estimate the parameters numerically.

8. ORDER STATISTICS

Let $Z_{1,}\ Z_2\ ...Z_n$ a MFW distributionmly from a MFW distribution with parameters $(\alpha,\gamma,\beta,\theta,\lambda)$, having ordered values

$Z_{1:n}, Z_{2:n}, \cdots, Z_{k:n}$. Let $Z_{(1:n)}$ positions the smallest of $\{Z_{1:n}, Z_{2:n}, \cdots, Z_{k:n}\}$, $Z_{(2:n)}$ positions the second smallest of $\{Z_{1:n}, Z_{2:n}, \cdots, Z_{k:n}\}$, similarly $Z_{(k:n)}$ positions the k^{th} smallest of $\{Z_{1:n}, Z_{2:n}, \cdots, Z_{k:n}\}$. The PDF of k^{th} order statistic is

$$g_{k:n}(z)=\frac{n!}{(k-1)!(n-k)!}g(z)\left[G(z)\right]^{k-1}\left[1-G(z)\right]^{n-k}\ .$$

So, the PDF of smallest order statistic is given by

$$g_{1:n}(z)=n\lambda\left(\gamma\beta z_1^{\gamma-1}+\alpha\theta z_1^{\alpha-1}\right)e^{\left(\beta z_1^{\gamma}\ +\ \theta z_1^{\alpha}\right)}\left(e^{-\lambda e^{\left(\beta z_1^{\gamma}\ +\ \theta z_1^{\alpha}\right)}}\right)^{n}.$$

PDF of largest order statistic is provided below

$$g_{k:n}(z)=n\lambda\left(\gamma\beta z_k^{\gamma-1}+\alpha\theta z_k^{\alpha-1}\right)e^{\left(\beta z_k^{\gamma}\ +\ \theta z_k^{\alpha}\right)}e^{-\lambda e^{\left(\beta z_k^{\gamma}\ +\ \theta z_k^{\alpha}\right)}}\left(1-e^{-\lambda e^{\left(\beta z_k^{\gamma}\ +\ \theta z_k^{\alpha}\right)}}\right)^{n}.$$

9. APPLICATION

In this section, a real life application is analyzed. A well-known data set, which is taken from Arset [1] is considered and the goodness of fit results of the MFW model is compared with that of eight other existing well-known lifetime models. Kolmogorov–Smirnov (K-S) test statistic, Akaike’s Information Criterion (AIC), Bayesian information criterion (BIC) and -log likelihood. On the basis these investigative measures, it is observed that the proposed model provides greater flexibility. The data set obtained from Arset [1], which represents the lifetimes of 50 devices. The data are provided in table 1 are summarized in table 2. The final results of the goodness of fit corresponding to the data given in example 1, are summarized in table 3.

Table 1: Life Time of 50 Devices

0.1, 0.2, 1, 1, 1, 1, 1, 2, 3, 6, 7, 1, 1, 12, 18, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 60, 63, 63, 67, 67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 84, 85, 85, 85, 85, 85, 86, 86

Table 2: Summary of the Arset Data

Min	1st Quartile	Median	Mean	3rd Quartile	Max
0.10	9.50	47.00	44.61	80.50	86.00

Table 3: Goodness of Fit Results for MFW Distribution and Other Eight Competing Models

Dist.	Max. Likelihood Estimates	Log-lik	K-S	AIC	BIC
MFW	$\hat{\alpha}$ **=0.13,** $\hat{\beta}$ **=1.06,** $\hat{\gamma}$ **=1.00,** $\hat{\theta}$ **=1.36,** $\hat{\lambda}$ **=0.96**	**-207.43**	**0.129**	**442.90**	**451.89**
FW	$\hat{\alpha}$ =0.246, $\hat{\beta}$ =2.867, $\hat{\gamma}$ =0.709, $\hat{\theta}$ =1.64	-211.78	0.148	449.65	458.43
FWEx	$\hat{\alpha}$ = 0.0122, $\hat{\beta}$ = 0.7002	-250.81	0.438	505.62	509.44
W	$\hat{\alpha}$ =44.913, $\hat{\beta}$ = 0.949	-241.00	0.239	486.00	489.82
EW	$\hat{\alpha}$ =91.023, $\hat{\beta}$ =4.69, $\hat{\sigma}$ = 0.164	-235.92	0.184	477.85	483.58
EFWEx	$\hat{\alpha}$ =0.0147, $\hat{\beta}$ =0.0147, $\hat{\theta}$ = 4.22	-226.98	0.143	459.97	465.71
EFWEx	$\hat{\alpha}$ =0.015, $\hat{\beta}$ =0.381, $\hat{\lambda}$ = 0.076	-224.83	0.158	455.66	461.40
TW	$\hat{\alpha}$ =0.83, $\hat{\beta}$ =0.05, $\hat{\lambda}$ =-0.29	-243.56	0.330	493.13	498.93
Ku-W	$\hat{\alpha}$ =0.92, $\hat{\beta}$ =0.008, $\hat{a}$ =0.85, $\hat{b}$ =3.00	-243.69	0.458	495.38	503.11

10. CONCLUSIONS

A new lifetime distribution, named as modified flexible Weibull distribution has been proposed and its properties are studied. The basic idea is to add an additional parameter to a flexible Weibull distribution, to introduce a more flexible statistical model. The proposed model accommodates bath-tub shaped failure rate, and offers greater distribution flexibility. It has shown that, the modified flexible Weibull distribution fits certain well-known lifetime data, set better than other eight existing modifications of the Weibull distribution. It is hoped that, the modified flexible Weibull distribution will serve as one of the most prominent lifetime distributions, and will attract a wide range of applications in reliability and biomedical analysis.

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